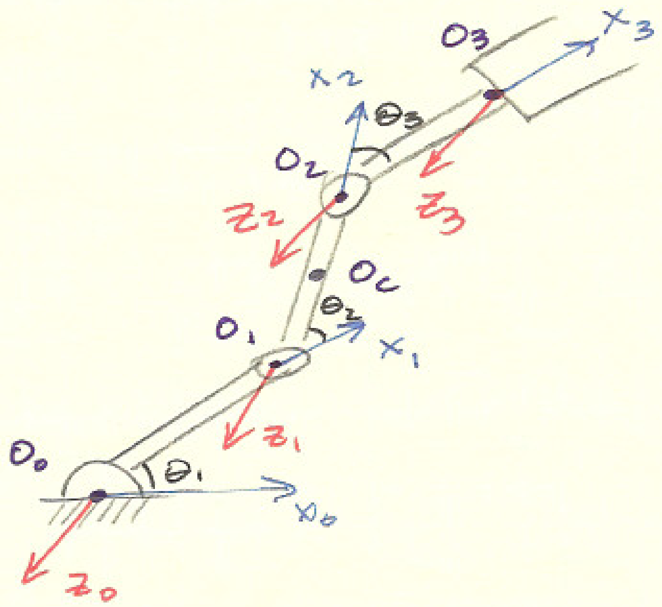
Ex:



Obtain the velocity at point O_c.

$$\begin{bmatrix} V_c \\ \omega_c \end{bmatrix} = \underbrace{J_c}_{2\times 3} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{z}_0 \times (O_c - O_0) & \hat{z}_1 & 0 \\ \hat{z}_0 & \hat{z}_1 & 0 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{z}_1 = \hat{z}_2$$

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad O_1 = \begin{bmatrix} a_1 C_1 \\ a_1 S_1 \\ 0 \end{bmatrix}$$

$$O_c = \begin{bmatrix} a_1 C_1 + \frac{a_2}{2} C_{12} \\ a_1 S_1 + \frac{a_2}{2} S_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \\ 0 \end{bmatrix}$$

$$O_c - O_1 = \begin{bmatrix} \frac{a_2}{2} C_{12} \\ \frac{a_2}{2} S_{12} \\ 0 \end{bmatrix}$$

$$\hat{z}_1 \times (O_c - O_1) = \begin{bmatrix} \hat{x}_0 & \hat{y}_0 & \hat{z}_0 \\ 0 & 0 & 1 \\ \frac{a_2}{2} C_{12} & \frac{a_2}{2} S_{12} & 0 \end{bmatrix}$$

$$= \hat{x}_0 \left(-\frac{a_2}{2} S_{12}\right) + \hat{y}_0 \left(+\frac{a_2}{2} C_{12}\right) + \hat{z}_0 (0)$$

$$= \begin{bmatrix} -\frac{a_2}{2} S_{12} \\ +\frac{a_2}{2} C_{12} \\ 0 \end{bmatrix}$$

$$\hat{z}_0 \times (O_c - O_0) = \begin{bmatrix} \hat{x}_0 & \hat{y}_0 & \hat{z}_0 \\ 0 & 0 & 1 \\ x_c & y_c & 0 \end{bmatrix}$$

$$= -\hat{x}_0 y_c - \hat{y}_0 (-x_c)$$

$$= \begin{bmatrix} -a_1 S_1 - \frac{a_2}{2} S_{12} \\ a_1 C_1 + \frac{a_2}{2} C_{12} \\ 0 \end{bmatrix}$$

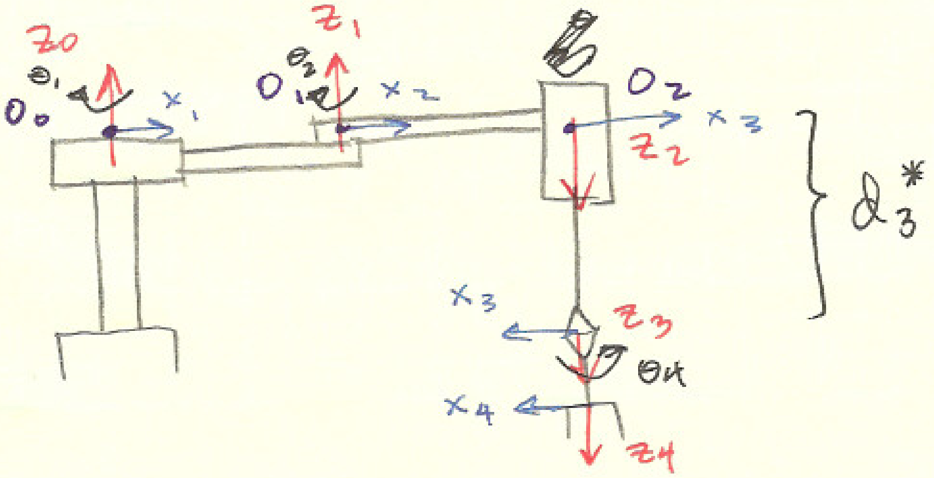
$$J_c = \begin{bmatrix} -Y_c & -\frac{a_2}{2} S_{12} & 0 \\ X_c & \frac{a_2}{2} C_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} V_c \\ \omega_c \end{bmatrix} = J_c \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$V_c = \begin{bmatrix} -y_c \dot{\theta}_1 - \frac{a_2}{2} S_{12} \dot{\theta}_{12} \\ x_c \dot{\theta}_1 + \frac{a_2}{2} C_{12} \dot{\theta}_2 \\ 0 \end{bmatrix}$$

$$\omega_c = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

Scara Manipulator



$$J = \begin{bmatrix} \hat{z}_0 \times (o_4 - o_0) & \hat{z}_1 \times (o_4 - o_1) & \hat{z}_2 & \hat{z}_3 \times (o_4 - o_3) \\ z_0 & \hat{z}_1 & 0 & \hat{z}_3 \end{bmatrix}$$

$$\hat{z}_3 \times (o_4 - o_3) = 0$$

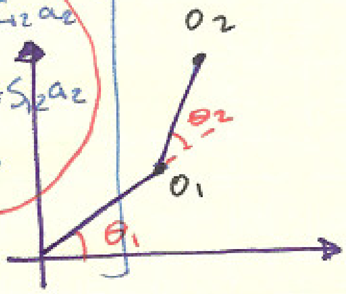
b/c they are parrellel.

$$\hat{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \hat{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \hat{z}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{z}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad \hat{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Top View

$$\begin{array}{c} \\ {}^0_2T = \end{array} \begin{array}{cccc} & & \hat{z}_2 & O_2 \\ \left[\begin{matrix} C_{12} \\ S_{12} \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} S_{12} \\ -C_{12} \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ -1 \\ 0 \end{matrix} & \left.\begin{matrix} a_1C_1 + C_{12}a_2 \\ a_1S_1 + S_{12}a_2 \\ 0 \\ 1 \end{matrix}\right] \end{array}$$



$$\begin{array}{c} \\ {}^0_3T = \end{array} \begin{array}{cccc} & & \hat{z}_3 & O_3 \\ \left[\begin{matrix} C_{12} \\ S_{12} \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} S_{12} \\ -C_{12} \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ -1 \\ 0 \end{matrix} & \left.\begin{matrix} a_1C_1 + a_2C_{12} \\ a_1S_1 + a_2S_{12} \\ -d_3 \\ 1 \end{matrix}\right] \end{array}$$

$$
\begin{array}{ccccc}
 & & & \hat{z}_3 & O_4 \\
{}^{0}_{4}T = & \left[\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{cc} C_4C_{12}+S_4S_{12} & -C_{12}S_4+S_{12}C_4 \\ S_{12}C_4-C_{12}S_4 & -S_4S_{12}-C_4C_{12} \\ 0 & 0 \\ 0 & 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ -1 \\ 0 \end{array} & \left.\begin{array}{c} a_1C_1+a_2C_{12} \\ a_1S_1+a_2S_{12} \\ -d_3-d_4 \\ 1 \end{array}\right]
\end{array}
$$

Note: It may be better to use a more straight forward method rather than calculating the full D-H repersentation.

$$
\hat{z}_0 \times (O_4 - O_0) = \begin{bmatrix} -a_1S_1 - a_2S_{12} \\ a_1C_1 + a_2C_{12} \\ 0 \end{bmatrix}
$$

$$
\hat{z}_1 \times (O_4 - O_1) = \begin{bmatrix} -a_2S_{12} \\ a_2C_{12} \\ 0 \end{bmatrix}
$$

$$
J = \begin{bmatrix} -a_1S_1 - a_2S_{12} & -a_2S_{12} & 0 & 0 \\ a_1C_1 + a_2C_{12} & a_2C_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}
$$

$$
\underset{6\times 1}{\begin{bmatrix} V_4 \\ \omega_4 \end{bmatrix}} = \underset{6\times 4}{J} \underset{4\times 1}{\begin{bmatrix} \dot{\theta}_1^* \\ \dot{\theta}_2^* \\ \dot{d}_3^* \\ \dot{\theta}_4^* \end{bmatrix}}
$$

Note: The problem with solving for the joint variables is inverting the Jacobian when it is not square this is why 6 DOF robots are so popular.

Singularities.

Consider the 2 link planar manipulator
The Jacobian is given by:

$$J = \begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \qquad \boxed{\begin{matrix} \dot{z} = 0 \\ \omega_x = 0 \\ \omega_y = 0 \end{matrix}}$$

$$\omega_z = \dot{\theta}_1 + \dot{\theta}_2$$

it is clear there is no single solution.

We can however write

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \end{bmatrix}}_{J_{xy}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Singular Configurations are given by $\det(J_{xy}) = 0$.